Memoir on the Algebra of Symbolic Logic.

By A. N. Whitehead, *Fellow of Trinity College, Cambridge, England.*

PART II.

The Theory of Substitutions.

§1.—*Types of Transformation.*

Any transformation of x and y into u and v can be represented by

$$x = f_1(u, v), \quad y = f_2(u, v). \tag{1}$$

Here $f_1(u, v)$ and $f_2(u, v)$ will be called the director functions of the transformation. They will always be represented by the following notation:

$$\left.\begin{aligned} f_1(u, v) &= a_1 uv + a_2 u\bar{v} + a_3 \bar{u}v + a_4 \bar{u}\bar{v}, \\ f_2(u, v) &= b_1 uv + b_2 u\bar{v} + b_3 \bar{u}v + b_4 \bar{u}\bar{v}. \end{aligned}\right\} \tag{2}$$

This transformation will also be called the transformation $\left.\begin{matrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \end{matrix}\right\}$, and $a_1, \ldots a_4;\ b_1, \ldots b_4$ will be called the coefficients of the transformation.

In general, x and y cannot be considered as independent variables when they are submitted to this transformation. For, by eliminating u and v from (1) and (2), we deduce

$$p(a_1, x;\ b_1, y)\, p(a_2, x;\ b_2, y)\, p(a_3, x;\ b_3, y)\, p(a_4, x;\ b_4, y) = 0. \tag{3}$$

Thus x and y are limited to be pairs of roots of equation (3). Conversely, equation (3) is the condition of the possibility of equations (1) and (2). Thus, if (3) is satisfied, (1) and (2) are satisfied. Hence, (1) and (2) form the general solution of (3).

Thus the theory of this type of transformation is simply the theory of equations. A few theorems connected with it will be given incidentally in this part.

A distinct type of transformation exists when equation (3) is satisfied identically for all values of x and y. The condition for this is that each of the coefficients of xy, $x\bar{y}$, $\bar{x}y$, $\bar{x}\bar{y}$ in the development of the left-hand side of (3) should vanish. This condition can be exhibited in the single fundamental equation

$$\Pi\,(\bar{a}_r + \bar{b}_r) + \Pi\,(\bar{a}_r + b_r) + \Pi\,(a_r + \bar{b}_r) + \Pi\,(a_r + b_r) = 0, \quad (r = 1,\ 2,\ 3,\ 4). \quad (4)$$

When this condition is satisfied, the transformation transforms the independent variables x and y into the independent variables u and v. Thus equations (1) and (2) can be solved for u and v in terms of x and y, and a reverse transformation is thus obtained of the form

$$u = F_1\,(x, y), \quad v = F_2\,(x, y), \qquad (5)$$

where

$$\left.\begin{aligned} F_1\,(x, y) &= \alpha_1 xy + \alpha_2 x\bar{y} + \alpha_3 \bar{x}y + \alpha_4 \bar{x}\bar{y}, \\ F_2\,(x, y) &= \beta_1 xy + \beta_2 x\bar{y} + \beta_3 \bar{x}y + \beta_4 \bar{x}\bar{y}. \end{aligned}\right\} \qquad (6)$$

Thus, this type of transformation is reversible. Let the term "substitution" be used exclusively for it.

In the case of substitutions, there is no advantage in changing the notations for the independent variables from x and y to u and v. Let a substitution T be defined to consist in the substitution of Tx for x and of Ty for y, where

$$\left.\begin{aligned} Tx &= a_1 xy + a_2 x\bar{y} + a_3 \bar{x}y + a_4 \bar{x}\bar{y}, \\ Ty &= b_1 xy + b_2 x\bar{y} + b_3 \bar{x}y + b_4 \bar{x}\bar{y}; \end{aligned}\right\} \qquad (7)$$

and the coefficients of the substitution T satisfy equation (4). Also $T\phi\,(x, y)$ is defined to mean $\phi\,(Tx, Ty)$: for instance, $T\bar{x}$ means $\overline{}(Tx)$.

§2.—*Relations between the Coefficients of a Substitution.*

We have now to consider more particularly the implications of equation (4). Let us first eliminate b_1, b_2, b_3, b_4 from the equation; put $\lambda\,(b_1, b_2, b_3, b_4)$ for the left-hand side. We find that the only factors composing $\Pi\lambda\begin{pmatrix} i, & i, & i, & i \\ 0, & 0, & 0, & 0 \end{pmatrix}$ which do not equal i are the six of the type $\lambda\,(0, 0, i, i)$, $\lambda\,(i, 0, 0, i)$, etc. Also,

$$\lambda\,(0, 0, i, i) = a_1 a_2 + \bar{a}_1 \bar{a}_2 + a_3 a_4 + \bar{a}_3 \bar{a}_4,$$

with analogous values for the other factors of the same type. Hence,

$$\Pi\lambda\begin{pmatrix} i, & i, & i, & i \\ 0, & 0, & 0, & 0 \end{pmatrix} = \Sigma a_p\, a_q\, a_r + \Sigma \bar{a}_p\, \bar{a}_q\, \bar{a}_r, \quad (p,\, q,\, r = 1,\ 2,\ 3,\ 4),$$

where, as usual, p, q, r are unequal in the same product.

Hence, the resultant of (4), after eliminating b_1, b_2, b_3, b_4, is

$$\Sigma a_p\, a_q\, a_r + \Sigma \bar{a}_p\, \bar{a}_q\, \bar{a}_r = 0, \quad (p,\, q,\, r) = 1,\ 2,\ 3,\ 4). \tag{8}$$

A similar equation holds of b_1, b_2, b_3, b_4. Hence, the director functions, Tx and Ty, of any substitution are each of deficiency two and of supplemental deficiency two. Thus, functions of this type play a fundamental rôle in the theory of substitutions.

The conditions (8), which hold between the coefficients of a function of deficiency two and supplemental deficiency two, can be put otherwise thus: We have from (8),

$$a_1\, a_2\, (a_3 + a_4) = 0,$$

hence,

$$a_1\, a_2 \mathrel{=\!\!\!\!\!\!\;\;\nleftarrow} \bar{a}_3\, \bar{a}_4.$$

Also

$$\bar{a}_3\, \bar{a}_4\, (\bar{a}_1 + \bar{a}_2) = 0,$$

hence,

$$\bar{a}_3\, \bar{a}_4 \mathrel{=\!\!\!\!\!\!\;\;\nleftarrow} a_1\, a_2.$$

Thus

$$a_1\, a_2 = \bar{a}_3\, \bar{a}_4.$$

Hence, equation (8) can be replaced by the set of equations

$$a_p\, a_q = \bar{a}_r\, \bar{a}_s, \quad (p,\, q,\, r,\, s = 1,\ 2,\ 3,\ 4). \tag{9}$$

A similar set of equations holds for b_1, b_2, b_3, b_4.

Secondly, these relations (8) or (9), between the coefficients of each of the director functions separately, do not exhaust the implications of equation (4). For, from equations (9), we deduce the two sets,

$$\left.\begin{aligned} \Sigma a_p\, a_q\, \bar{b}_r\, \bar{b}_s &= \Sigma \bar{a}_p\, \bar{a}_q\, b_r\, b_s = \Sigma a_p\, a_q\, b_p\, b_q = \Sigma \bar{a}_p\, \bar{a}_q\, \bar{b}_p\, \bar{b}_q, \\ \Sigma a_p\, a_q\, b_r\, b_s &= \Sigma a_p\, a_q\, \bar{b}_p\, \bar{b}_q = \Sigma \bar{a}_p\, \bar{a}_q\, \bar{b}_r\, \bar{b}_s = \Sigma \bar{a}_p\, \bar{a}_q\, b_p\, b_q; \end{aligned}\right\} \tag{10}$$

where $(p,\, q,\, r,\, s = 1,\ 2,\ 3,\ 4)$.

Then, by the application to equation (4) of (8) and (10), we deduce

$$\Sigma a_p\, a_q\, b_p\, b_q + \Sigma \bar{a}_p\, \bar{a}_q\, b_p\, b_q = 0, \tag{11}$$

or equivalent forms deduced by the use of (10). A complete equivalent to equation (4) is given by

$$\Sigma\,(a_p a_q + \bar{a}_p \bar{a}_q)(b_p b_q + \bar{b}_p \bar{b}_q) = 0\,, \quad (p,\, q = 1,\, 2,\, 3,\, 4)\,; \tag{12}$$

that is,

$$\Sigma \bar{p}\,(a_p,\, a_q\,;\; b_p,\, b_q) = 0\,.$$

It is easy to see that this equation includes (11). It also includes (8), for we can deduce (8) from it by eliminating b_1, b_2, b_3, b_4.

Equation (8) may be solved for a_3 and a_4. The general solution is

$$\left.\begin{aligned} a_3 &= \bar{a}_1 \bar{a}_2 + p\,(\bar{a}_1 + \bar{a}_2)\,, \\ a_4 &= \bar{a}_1 \bar{a}_2 + \bar{p}\,(\bar{a}_1 + \bar{a}_2)\,. \\ \text{Similarly,} \qquad b_3 &= \bar{b}_1 \bar{b}_2 + q\,(\bar{b}_1 + \bar{b}_2)\,, \\ b_4 &= \bar{b}_1 \bar{b}_2 + \bar{q}\,(\bar{b}_1 + \bar{b}_2)\,. \end{aligned}\right\} \tag{13}$$

But a_1, a_2, b_1, b_2, and p and q, cannot be assumed arbitrarily consistently with the equation (4). For from (12) we find

$$(a_1 a_2 + \bar{a}_1 \bar{a}_2)(b_1 b_2 + \bar{b}_1 \bar{b}_2) = 0 \tag{14}$$

and

$$(a_1 \bar{a}_2 b_1 \bar{b}_2 + \bar{a}_1 a_2 \bar{b}_1 b_2)(pq + \bar{p}\bar{q}) + (a_1 \bar{a}_2 \bar{b}_1 b_2 + \bar{a}_1 a_2 b_1 \bar{b}_2)(p\bar{q} + \bar{p}q) = 0\,. \tag{15}$$

Equation (15) does not require any relation to be fulfilled between a_1, a_2, b_1, b_2. Thus when a_1, a_2, a_3, a_4 have been chosen to satisfy (14), p and q can be chosen to satisfy (15), and then a_3, a_4, b_3, b_4 are given by equation (13). By this process the coefficients of any substitution can be found.

It is to be noticed that any three of a_1, a_2, b_1, b_2 can be chosen arbitrarily, and equation (14) can then be satisfied by a proper choice of the fourth. Also, either p or q can be assumed arbitrarily and equation (15) satisfied by a proper choice of the other.

Again, if a_1, a_2, a_3 have been chosen so as to satisfy

$$a_1 a_2 a_3 + \bar{a}_1 \bar{a}_2 \bar{a}_3 = 0\,, \tag{m}$$

then a_4 is definitely determined by the equation

$$a_4 = \bar{a}_2 \bar{a}_3 + \bar{a}_3 \bar{a}_1 + \bar{a}_1 \bar{a}_2\,. \tag{16}$$

For, solve the first of equations (13) to find the general value of p which is con-

sistent with the given values of a_1, a_2, a_3 and simplify by the use of (m). We find

$$p = a_1 a_3 + a_2 a_3 + u(a_3 + a_1 a_2).$$

Thence, from the second of equations (13), we find

$$a_4 = \bar{a}_1 \bar{a}_2 + \bar{a}_1 \bar{a}_3 + \bar{a}_2 \bar{a}_3 + u(\bar{a}_1 \bar{a}_3 + \bar{a}_2 \bar{a}_3 + \bar{a}_1 \bar{a}_2),$$

which reduces to equation (16). Hence, if three coefficients of a function $\phi(x, y)$ of deficiency two and of supplemental deficiency two are given, the function is completely determined. Thus, if the six coefficients, a_1, a_2, a_3, b_1, b_2, b_3 of a substitution are given, the substitution is completely determined.

§3.—*The Reverse Substitution.*

There is only one reverse substitution corresponding to a given substitution T; that is, there is only one substitution T' such that

$$T'Tx = x, \quad T'Ty = y.$$

This proposition requires proof, for if we solve the equations

$$x = f_1(u, v), \quad y = f_2(u, v),$$

for u and v we find

$$u = F_1(x, y, q_1, q_2), \quad v = F_2(x, y, q_1, q_2),$$

where q_1 and q_2 are arbitraries which have been introduced in the solution, and each particular choice of q_1 and q_2 would seem to determine a different substitution which is reverse to T. We have to show that there is only one set of roots for u and v in terms of x and y.

The equations for u and v can be written

$$p(a_1, x)uv + p(a_2, x)u\bar{v} + p(a_3, x)\bar{u}v + p(a_4, x)\bar{u}\bar{v} = 0,$$
$$p(b_1, y)uv + p(b_2, y)u\bar{v} + p(b_3, y)\bar{u}v + p(b_4, y)\bar{u}\bar{v} = 0.$$

These two equations can be combined into

$$p(a_1, x; b_1, y)uv + p(a_2, x; b_2, y)u\bar{v} + p(a_3, x; b_3, y)\bar{u}v + p(a_4, x; b_4, y)\bar{u}\bar{v} = 0.$$

The necessary and sufficient condition (cf. Part I, §3) that this equation for u and v should only have one set of roots is that the left-hand side should be a secondary linear prime. The condition for this is

$$\Sigma \bar{p}(a_q, x; b_q, y)\bar{p}(a_r, x; b_r, y) = 0, \quad (q, r = 1, 2, 3, 4).$$

This equation can be written

$$\Sigma\,(a_q a_r x + \bar{a}_q \bar{a}_r \bar{x})(b_q b_r y + \bar{b}_q \bar{b}_r \bar{y}) = 0, \quad (q, r = 1, 2, 3, 4).$$

But from equation (12), this equation is satisfied identically for all values of x and y. Thus there is only one substitution reverse to T. Let it be written T^{-1}. It is evident that T is the reverse substitution to T^{-1}. Also, let the identical substitution be written T^0, so that

$$TT^{-1} = T^{-1}T = T^0.$$

Assume that

$$T^{-1}x = \alpha_1 xy + \alpha_2 xy + \alpha_3 xy + \alpha_4 \bar{x}\,\bar{y},$$
$$T^{-1}y = \beta_1 xy + \beta_2 x\bar{y} + \beta_3 xy + \beta_4 \bar{x}\,\bar{y}.$$

Then, by hypothesis,

$$T^{-1}f_1(x, y) = x, \quad T^{-1}f_2(x, y) = y.$$

Hence, for all values of x and y,

$$x = f_1(\alpha_1, \beta_1)\,xy + f_1(\alpha_2, \beta_2)\,x\bar{y} + f_1(\alpha_3, \beta_3)\,\bar{x}\,y + f_1(\alpha_4, \beta_4)\,\bar{x}\,\bar{y},$$
$$y = f_2(\alpha_1, \beta_1)\,xy + f_2(\alpha_2, \beta_2)\,x\bar{y} + f_2(\alpha_3, \beta_3)\,\bar{x}\,y + f_2(\alpha_4, \beta_4)\,\bar{x}\,\bar{y}.$$

Hence,

$$\bar{f_1}(\alpha_1, \beta_1) = 0, \quad \bar{f_1}(\alpha_2, \beta_2) = 0, \quad f_1(\alpha_3, \beta_3) = 0, \quad f_1(\alpha_4, \beta_4) = 0,$$
$$\bar{f_2}(\alpha_1, \beta_1) = 0, \quad f_2(\alpha_2, \beta_2) = 0, \quad \bar{f_2}(\alpha_3, \beta_3) = 0, \quad f_2(\alpha_4, \beta_4) = 0.$$

Hence, α_1 and β_1 satisfy the equation

$$\bar{f_1}(u, v) + \bar{f_2}(u, v) = 0;$$

and α_2 and β_2 satisfy the equation

$$\bar{f_1}(u, v) + f_2(u, v) = 0;$$

and α_3 and β_2 satisfy the equation

$$f_1(u, v) + \bar{f_2}(u, v) = 0;$$

and α_4 and β_4 satisfy the equation

$$f_1(u, v) + f_2(u, v) = 0.$$

By the use of equations (8) and (10) and (11), it is easy to verify that the left-hand side of each of these equations is a secondary linear prime (cf. Part I, equation (4)) and, therefore, each equation has only one set of roots. Thus we obtain another proof of the previous proposition. Again, if $\phi(x, y)$ is a secondary linear prime, the solution of $\phi(x, y) = 0$ is

$$x = CD, \quad y = BD.$$

Hence, the solution for the coefficients of the reverse substitution are

$$\left.\begin{array}{ll} \alpha_1 = (\bar{a}_3 + \bar{b}_3)(\bar{a}_4 + \bar{b}_4), & \beta_1 = (\bar{a}_2 + \bar{b}_2)(\bar{a}_4 + \bar{b}_4), \\ \alpha_2 = (\bar{a}_3 + b_3)(\bar{a}_4 + b_4), & \beta_2 = (\bar{a}_2 + b_2)(\bar{a}_4 + b_4), \\ \alpha_3 = (a_3 + \bar{b}_3)(a_4 + \bar{b}_4), & \beta_3 = (a_2 + \bar{b}_2)(a_4 + \bar{b}_4), \\ \alpha_4 = (a_3 + b_3)(a_4 + b_4), & \beta_4 = (a_2 + b_2)(a_4 + b_4) \end{array}\right\} \tag{17}$$

An interesting example of a substitution is

$$\left.\begin{array}{l} Tx = p\,(a, x), \\ Ty = p\,(b, y). \end{array}\right\} \tag{18}$$

It is easy to verify that in this case

$$T = T^{-1}, \text{ that is, } T^2 = T^0.$$

§4.—*The Group of Substitutions.*

If T_1 and T_2 are any two substitutions, then $T_1\,T_2$ and $T_2\,T_1$ are substitutions (as distinct from non-reversible transformations), also we have proved that one, and only one, reverse substitution corresponds to any given substitution. Hence, substitutions form a group.

The group of substitutions is not continuous since the concepts of quantity and of infinitesimal variations of quantities have no place among the concepts of this algebra. It is of finite order, if the number of distinct terms in the algebra representing given fundamental constants from which all reasoning starts, is conceived as finite. It is of indefinite order in so far as we may always suppose new constants to be produced without violating any of the laws of the algebra.

If T be any substitution the subgroup T, T^2, T^3, ... is necessarily of finite order, and is therefore, cyclical. For only a finite number of terms can be generated by algebraic combinations of the coefficients of T, and the coefficients of $T^2\,T^3$, ... must be selections from these coefficients. The order can be reduced by the assumption of additional relations among the coefficients of T. Thus, in the substitution of equation (18), the order is two.

§5.—*Substitutions Satisfying Special Conditions.*

To find the condition which the functions $\psi_1(x, y)$, $\psi_2(x, y)$, $\psi_3(x, y)$, $\psi_4(x, y)$, must satisfy, if the coefficients of some substitution T are such that, with the

usual notation, a_1 and b_1 are a pair of roots of $\psi_1(x, y)$, a_2 and b_2 of $\psi_2(x, y)$, a_3 and b_3 of $\psi_3(x, y)$, a_4 and b_4 of $\psi_4(x, y)$. These equations can be expressed in the typical form

$$\psi_r(a_r, b_r) = 0, \quad (r = 1, 2, 3, 4) \ldots. \qquad (19)$$

Let $$\psi_r(x, y) = F_r xy + G_r x\bar{y} + H_r \bar{x}y + K_r \bar{x}\bar{y}.$$

The complete condition satisfied by a_r, b_r, $(r = 1, 2, 3, 4)$ is found by combining equations (4) and (19). It becomes.

$$\Pi(\bar{a}_r + \bar{b}_r) + \Pi(\bar{a}_r + b_r) + \Pi(a_r + \bar{b}_r) + \Pi(a_r + b_r) + \Sigma\psi_r(a_r, b_r) = 0, \quad (20)$$
$$(r = 1, 2, 3, 4).$$

We have to eliminate a_r, b_r $(r = 1, 2, 3, 4)$ from this equation. Put $\lambda(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ for its left-hand side. Now,

$$\begin{array}{llll}
\Pi(\bar{a}_r + \bar{b}_r) = i, & \text{unless at least one pair,} & a_r = i, & b_r = i, \\
\Pi(\bar{a}_r + b_r) = i, & \text{" \quad " \quad " \quad "} & a_r = i, & b_r = 0, \\
\Pi(a_r + \bar{b}_r) = i, & \text{" \quad " \quad " \quad "} & a_r = 0, & b_r = i, \\
\Pi(a_r + b_r) = i, & \text{" \quad " \quad " \quad "} & a_r = 0, & b_r = 0,
\end{array}$$

Hence, each of the factors composing $\Pi\lambda\begin{pmatrix} i, & i, & i, & i, & i, & i, & i, & i \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \end{pmatrix}$ is i, except those in which simultaneously $a_p = i$, $b_p = i$; $a_q = i$, $b_q = 0$; $a_r = 0$, $b_r = i$, $a_s = 0$, $b_s = 0$; where $(p, q, r, s = 1, 2, 3, 4)$.

Hence the resultant of equation (20) is

$$\Pi(F_p + G_q + H_r + K_s) = 0, \quad (p, q, r, s = 1, 2, 3, 4). \qquad (21)$$

This is the required condition which the coefficients of equations (19) must satisfy.

We can deduce as special cases of the above, the following theorems:

The condition which the coefficients of the first three of equations (19) must satisfy in order that (a_1, b_1) may be chosen to satisfy the first, (a_2, b_2) the second, and (a_3, b_3) the third, is deduced from equation (21) by putting zero for F_4, G_4, H_4, K_4.

The condition which the coefficients of the first two of equation (19) must satisfy in order that (a_1, b_1) may be chosen to satisfy the first and (a_2, b_2) the second, is found from equation (21) by putting zero for F_3, G_3, H_3, K_3 and also

for F_4, G_4, H_4, K_4. It can be written out in full in the form

$$F_1 G_1 H_1 K_1 + F_2 G_2 H_2 K_2 + F_1 G_1 H_1 . F_2 G_2 H_2 + F_1 G_1 K_1 . F_2 G_2 K_2 + F_1 H_1 K_1 . F_2 H_2 K_2 + G_1 H_1 K_1 . G_2 H_2 K_2 = 0. \quad (22)$$

This condition in addition to securing that the two equations

$$\psi_1 (x, y) = 0, \ \psi_2 (x y) = 0,$$

are possible, also secures that the function $\psi_1 (x, y) \ \psi_2 (x, y)$ is of supplemental deficiency two. This latter part of the condition is satisfied if either of the functions is of supplemental deficiency two. For instance both (a_1, b_1) and (a_2, b_2) can be chosen to be pairs of roots of the same equation

$$\psi_1 (x, y) = 0,$$

if $\psi_1 (x, y)$ is of supplemental deficiency two. Similarly from the previous theorem it follows that (a_1, b_1), (a_2, b_2), (a_3, b_3) can be chosen to be pairs of roots of the same equation, $\psi_1 (x, y) = 0$, if $\psi_1 (x, y)$ is of supplemental deficiency three. But it follows from (21) that it is impossible that (a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4) should be pairs of roots of the same equation.

This fact, that there are sets of four pairs of terms which cannot be severally pairs of roots of the same equation of two variables, is one of the fundamental facts of the algebra. The analogue for equations of one variable is that the two terms a and $\overline{a}$ cannot be both roots of the same equation of one variable. This fundamental fact is the correlative of the fact that there are functions which are not evanescible.

Again, if we assume that a_2 and b_2 are to satisfy the second of equations (19) and a_3 and b_3 the third, and a_4 and b_4 the fourth of equations (19), let us investigate the conditions which are thereby imposed on a_1 and b_1. We have to eliminate a_2, b_2, a_3, b_3, a_4, b_4, from these three equations and from equation (4); and the resulting equation for a_1, b_1 will be the required condition. It will be more convenient to substitute the equivalent equation (12) for equation (4). Thus the complete equation from which the elimination is to be performed can be written

$$\Sigma (a_1 a_p + \overline{a}_1 \overline{a}_p) (b_1 b_p + \overline{b}_1 \overline{b}_p) + \Sigma (a_p a_q + \overline{a}_p \overline{a}_q) (b_p b_q + \overline{b}_p \overline{b}_q) + \Sigma \psi_p (a_p, b_p) = 0, \quad (p, q = 2, 3, 4).$$

Now put $\lambda (a_2, a_3, a_4, b_2, b_3, b_4)$ for the left-hand side of this equation. Then any of the factors composing $\Pi\lambda \begin{pmatrix} i, & i, & i, & i, & i, & i \\ 0, & 0, & 0, & 0, & 0, & 0 \end{pmatrix}$, for which either $a_2 = a_3$, and $b_2 = b_3$

simultaneously, or $a_2 = a_4$ and $b_2 = b_4$ simultaneously, or $a_3 = a_4$ and $b_3 = b_4$ simultaneously, it has the value i. Accordingly considering the factors which involve none of these three possibilities, we find that the required condition is

$$\Pi\,(G_p + H_q + K_r)\,a_1 b_1 + \Pi\,(F_p + H_q + K_r)\,a_1 \overline{b_1} + \Pi\,(F_p + G_q + K_r)\,\overline{a_1}\,b_1 + \Pi\,(F_p + G_q + H_r)\,\overline{a_1}\,\overline{b_1} = 0,\ (p,\,q,\,r = 2, 3, 4). \quad (23)$$

Thus if $a_1 b_1$ is also to satisfy the first of equations (19), the complete condition which it satisfies is

$$\{F_1 + \Pi\,(G_p + H_q + K_r)\}\,a_1 b_1 + \{G_1 + \Pi\,(F_p + H_q + K_r)\}\,a_1\overline{b_1} + \{H_1 + \Pi\,(F_p + G_q + K_r)\}\,\overline{a_1} b_1 + \{K_1 + \Pi\,(F_p + G_q + H_r)\}\,\overline{a_1}\overline{b_1} = 0. \quad (24)$$

The condition for the possibility of this equation is simply equation (21), as should evidently be the case.

§6.—*Congruence of Functions.*

Two functions $\phi\,(x, y)$ and $\Phi\,(x, y)$ are said to be congruent, if one or more substitutions T exist, such that

$$T\phi\,(x, y) = \Phi\,(x, y).$$

The relation of congruence will be expressed by the symbolism

$$\phi\,(x, y) \longleftrightarrow \Phi\,(x, y). \quad (25)$$

It is evident that if

$$\phi\,(x, y) \longleftrightarrow \Phi\,(x, y),$$

and

$$\Phi\,(x, y) \longleftrightarrow \psi\,(x, y),$$

then

$$\phi\,(x, y) \longleftrightarrow \psi\,(x, y).$$

Thus the relation is transitive.

A set of functions such that any two are congruent, and which comprises all the functions congruent to members of the set, is called a congruent family.

We will now prove the fundamental theorem that a congruent family is composed of all the functions with a given set of invariants.

For let

$$\phi\,(x, y) = Axy + Bx\overline{y} + C\overline{x}y + D\overline{x}\,\overline{y},$$
$$\Phi\,(x, y) = Fxy + Gx\overline{y} + H\overline{x}y + K\overline{x}\,\overline{y}.$$

We require the necessary and sufficient condition that

$$\phi\,(x, y) \longleftrightarrow \Phi\,(x, y).$$

Now with the usual notation for the coefficients of the substitution T, we have

$$T\phi(x, y) = \phi(a_1, b_1)\,xy + \phi(a_2, b_2)\,x\overline{y} + \phi(a_3, b_3)\,\overline{x}y + \phi(a_4, b_4)\,\overline{x}\,\overline{y}.$$

Hence $\phi(a_1, b_1) = F,\ \phi(a_2, b_2) = G,\ \phi(a_3, b_3) = H,\ \phi(a_4, b_4) = K.$
These equations can be written

$$\left.\begin{array}{l}
p(A, F)\,a_1b_1 + p(B, F)\,a_1\overline{b}_1 + p(C, F)\,\overline{a}_1b_1 + p(D, F)\,\overline{a}_1\overline{b}_1 = 0, \\
p(A, G)\,a_2b_2 + p(B, G)\,a_2\overline{b}_2 + p(C, G)\,\overline{a}_2b_2 + p(D, G)\,\overline{a}_2\overline{b}_2 = 0, \\
p(A, H)\,a_3b_3 + p(B, H)\,a_3\overline{b}_3 + p(C, H)\,\overline{a}_3b_3 + p(D, H)\,\overline{a}_3\overline{b}_3 = 0, \\
p(A, K)\,a_4b_4 + p(B, K)\,a_4\overline{b}_4 + p(C, K)\,\overline{a}_4b_4 + p(D, K)\,\overline{a}_4\overline{b}_4 = 0.
\end{array}\right\} \quad (26)$$

Now by comparison with equations (19) and (21), we see that the requisite condition is

$$\Pi p(A, F;\ B, G;\ C, H;\ D, K) = 0, \qquad (27)$$

where, in order to obtain the various factors, F, G, H, K are kept in the same positions, and A, B, C, D are permuted in every possible way, so that each appears in each factor.

In order to effect the solution of this equation, it will be convenient to alter the notation. Consider

$$\Pi p(x_1, c_p;\ x_2, c_q;\ x_3, c_r;\ x_4, c_s) = 0, \quad (p, q, r, s = 1, 2, 3, 4). \qquad (28)$$

Let C_1, C_2, C_3, C_4 be the symmmetric functions of c_1, c_2, c_3, c_4; and let X_1, X_2, X_3, X_4 be the symmetric functions of x_1, x_2, x_3, x_4.
Now, consider any one factor of the left-hand side of (28); for example, $p(x_1, c_p;\ x_2, c_q;\ x_3, c_r;\ x_4, c_s)$.
Then

$$p(x_1, c_p;\ x_2, c_q;\ x_3, c_r;\ x_4, c_s) = \overline{C}_4x_1x_2x_3x_4 + \Sigma\left[(\overline{c}_p + \overline{c}_q + \overline{c}_r + c_s)\,x_1x_2x_3\overline{x}_4\right]$$
$$+ \Sigma\left[(\overline{c}_p + \overline{c}_q + c_r + c_s)\,x_1x_2\overline{x}_3\overline{x}_4\right] + \Sigma\left[(\overline{c}_p + c_q + c_r + c_s)\,x_1\overline{x}_2\overline{x}_3\overline{x}_4\right] + C_1\overline{x}_1\overline{x}_2\overline{x}_9\overline{x}_4.$$

Hence, equation (28) becomes

$$\overline{C}_4X_4 + \overline{\ }(C_3\overline{C}_4)\,X_3\overline{X}_4 + \overline{\ }(C_2\overline{C}_3)\,X_2\overline{X}_3 + \overline{\ }(C_1\overline{C}_2)\,X_1\overline{X}_2 + C_1\overline{X}_1 = 0.$$

But, from the symmetry of (28), it is obvious that we may equally well write

$$\overline{X}_4C_4 + \overline{\ }(X_3\overline{X}_4)\,C_3\overline{C}_4 + \overline{\ }(X_2\overline{X}_3)\,C_2\overline{C}_3 + \overline{\ }(X_1\overline{X}_2)\,C_1\overline{C}_2 + X_1\overline{C}_1 = 0.$$

By adding these two equations, and remembering that

$$P\overline{Q} + \overline{P}Q = 0,$$

implies $P = Q$, we find

$$X_4 = C_4, \; X_3\overline{X_4} = C_3\overline{C_4}, \; X_2\overline{X_3} = C_2\overline{C_3}, \; X_1\overline{X_2} = C_1\overline{C_2}, \; X_1 = C_1. \qquad \text{(m)}$$

The fourth and fifth of equations (m) give

$$C_1\overline{X_2} = C_1\overline{C_2}.$$

Hence,
$$X_2 = C_2 + q\overline{C_1}.$$

But
$$X_2\overline{C_1} = X_2\overline{X_1} = 0,$$

and
$$C_2\overline{C_1} = 0.$$

Hence,
$$q\overline{C_1} = 0.$$

Thus $X_2 = C_2$. An exactly similar proof applied to the third of equations (m) will now prove that $X_3 = C_3$.

Hence, equation (27) is equivalent to the fact that the corresponding invariants of $\phi(x, y)$ and $\Phi(x, y)$ are equal.

As special examples of this theorem, we note that (1) secondary linear primes form a congruent family, (2) secondary separable primes form a congruent family, (3) functions both of deficiency two and of supplemental deficiency two form a congruent family. This last family includes as members all primary primes whether functions of x only or of y only.

A congruent family is entirely defined by its invariants. Accordingly, we shall name a family by its invariants, and speak of the family (S_1, S_2, S_3, S_4).

Any family (S_1, S_2, S_3, S_4) includes as a member the function

$$S_1xy + S_2x\bar{y} + S_3\bar{x}y + S_4\bar{x}\bar{y},$$

which can also be written

$$S_1xy + S_2x + S_3y + S_4;$$

and also the family includes the twenty-three other functions of the same type found by permuting the invariants in their use as coefficients.

Call such functions the "canonical functions" of the family.

There is no fundamental distinction in property between the various canonical functions of a family. Accordingly, we shall habitually use the one mentioned above, and will call it "the canonical function." Thus the canonical function of the family $(i, i, i, 0)$, which is the family of secondary linear primes, is $x + y$. The canonical function of the family $(i, 0, 0, 0)$ which is the family of secondary separable primes, is xy. The canonical function of the family

$(i, i, 0, 0)$ is x. The only families which contain some linear functions as some members, have been proved (cf. Part I, §9) to be those for which $S_1 = S_2$. Thus the canonical function for such a family is $S_1x + S_3y + S_4$. Similarly, the only families which contain some separable functions as some members are those for for which $S_3 = S_4$. Thus the canonical function for such a family is

$$S_1xy + S_2x + S_4, \text{ that is, } (x + S_4)(S_1y + S_2).$$

Also, by reference to equations (23) and (24), we can find from equations (26) the complete conditions satisfied by a_1 and b_1 after the other coefficients have been eliminated. For let S_{11}, S_{12}, S_{13} be the symmetric functions of B, C, D; and let S_{21}, S_{22}, S_{23} be the symmetric functions of A, C, D, and S_{31}, S_{32}, S_{33} of A, B, D and S_{41}, S_{42}, S_{43} of A, B, C; and let R_{11}, R_{12}, R_{13} be the symmetric functions of G, H, K, and so on. Now, consider $\Pi p(B, G; C, H; D, K)$, where the various factors are formed by keeping G, H, K in their places and permuting B, C, D. By comparison with the evaluation of the left-hand side of (28), we see that

$$\begin{aligned}&\Pi p(B, G; C, H; D, K)\\ &\qquad = p(S_{13}, R_{13}; S_{12}\overline{S_{13}}, R_{12}\overline{R_{13}}; S_{11}\overline{S_{12}}, R_{11}\overline{R_{12}}; S_{11}, R_{11}),\end{aligned}$$

with similar expressions for other similar products. Thus we find the required equation to be

$$\begin{aligned}&p(A, F; S_{13}, R_{13}; S_{12}\overline{S_{13}}, R_{12}\overline{R_{13}}; S_{11}\overline{S_{12}}, R_{11}\overline{R_{12}}; S_{11}, R_{11})\,a_1b_1\\ &\quad + p(B, F; S_{23}, R_{13}; S_{22}\overline{S_{23}}, R_{12}\overline{R_{13}}; S_{21}\overline{S_{22}}, R_{11}\overline{R_{12}}; S_{21}, R_{11})\,a_1\overline{b_1}\\ &\quad + p(C, F; S_{33}, R_{13}; S_{32}\overline{S_{33}}, R_{12}\overline{R_{13}}; S_{31}\overline{S_{32}}, R_{11}\overline{R_{12}}; S_{31}, R_{11})\,\overline{a_1}b_1\\ &\quad + p(D, F; S_{43}, R_{13}; S_{42}\overline{S_{43}}, R_{12}\overline{R_{13}}; S_{41}\overline{S_{42}}, R_{11}\overline{R_{12}}; S_{41}, R_{11})\,\overline{a_1}\overline{b_1} = 0. \quad (29)\end{aligned}$$

Also a similar equation can be found for a_2, b_2 by putting G for F and R_{21}, R_{22}, R_{23} for R_{11}, R_{12}, R_{13} respectively in equation (29). Also, similarly for a_3, b_3 and for a_4, b_4 by similar interchanges.

Thus a_1, b_1 can be chosen to be any pair of roots of equation (29), but when a_1, b_1 is once chosen, a_2, b_2 and a_3, b_3 and a_4, b_4 must be suitable pairs of roots of their corresponding equations.

It is easily proved by considering the invariants, that if

$$\phi_1(x, y) \longleftrightarrow \Phi_1(x, y)$$

and

$$\phi_2(x, y) \longleftrightarrow \Phi_2(x, y),$$

then, for all values of λ,

$$\lambda\phi_1(x, y) + \bar{\lambda}\phi_2(x, y) \Longleftrightarrow \lambda\Phi_1(x, y) + \bar{\lambda}\Phi_2(x, y).$$

The conditions that $\phi(x, y)$ can be transformed into $\Phi(x, y)$ by some transformation which is not necessarily a substitution, are given by the resultants of equations (26) without the use of equation (4). These conditions are, if S_1, S_2, S_3, S_4 are the invariants of $\phi(x, y)$ and R_1, R_2, R_3, R_4 of $\Phi(x, y)$,

$$\bar{S}_1 F + S_4\bar{F} = 0, \quad \bar{S}_1 G + S_4\bar{G} = 0, \quad \bar{S}_k H + S_4\bar{H} = 0,$$
$$\bar{S}_1 K + S_4\bar{K} = 0.$$

Hence, by addition,

$$\bar{S}_1 R_1 + S_4\bar{R}_4 = 0.$$

Thus the required conditions can be written

$$R_1 \mathrel{=\!\!(} S_1, \quad S_4 \mathrel{=\!\!(} R_4. \tag{30}$$

In other words, the field of $\phi(x, y)$ must contain the field* of $\Phi(x, y)$. Thus the conditions that $\phi(x, y)$ can be transformed into $\Phi(x, y)$, and that $\Phi(x, y)$ can be transformed into $\phi(x, y)$, are

$$R_1 = S_1, \quad R_4 = S_4. \tag{31}$$

Accordingly, the additional conditions required in order that these transformations may be substitutions, are

$$R_2 = S_2, \quad R_3 = S_2.$$

§7.—*The Identical Group of a Function.*

Any function $\phi(x, y)$ can be conceived as congruent to itself. Also the substitutions such that

$$T\phi(x, y) = \phi(x, y)$$

evidently form a group. For, if T is such a substitution, T^{-1} also has the same property, and if T_1 is another such substitution, then $T_1 T$ has also the same property. Let this group be called the "identical group" of the function $\phi(x, y)$.

---

* Cf. "Universal Algebra," §33.

By substituting A, B, C, D for F, G, H, K in equations (26), we see that the coefficients of a substitution of the identical group must satisfy the equations

$$\left.\begin{array}{l} * \ + p(B, A)\, a_1 \bar{b}_1 + p(C, A)\, \bar{a}_1 b_1 + p(D, A)\, \bar{a}_1 \bar{b}_1 = 0, \\ p(A, B)\, a_2 b_2 + \ * \ + p(C, B)\, \bar{a}_2 b_2 + p(D, B)\, \bar{a}_2 \bar{b}_2 = 0, \\ p(A, C)\, a_3 b_3 + p(B, C)\, a_3 \bar{b}_3 + \ * \ + p(D, C)\, \bar{a}_3 \bar{b}_3 = 0, \\ p(A. D)\, a_4 b_4) p(B, D)\, a_4 \bar{b}_4 + p(C, D)\, \bar{a}_4 b_4 + \ * \ = 0, \end{array}\right\} \qquad (32)$$

The coefficients must also, of course, satisfy equation (4).

Then equation (29), in the reduced form required for substitutions of the identical group of $\phi(x, y)$, becomes

$$\begin{array}{l} p(B, A;\ S_{23}, S_{13};\ S_{22} \bar{S}_{23}, S_{12} \bar{S}_{13};\ S_{21} \bar{S}_{22}, S_{11} \bar{S}_{12};\ S_{21}, S_{11})\, \bar{a}_1 b_1 \\ \quad + p(C, A;\ S_{33}, S_{13};\ S_{32} \bar{S}_{33}, S_{12} \bar{S}_{13};\ S_{31} \bar{S}_{32}, S_{11} \bar{S}_{12};\ S_{31}, S_{11})\, a_1 \bar{b}_1 \\ \quad + p(D, A;\ S_{43}, S_{13};\ S_{42} \bar{S}_{43}, S_{12} \bar{S}_{13};\ S_{41} \bar{S}_{42}, S_{11} \bar{S}_{12};\ S_{41}, S_{11})\, \bar{a}_1 \bar{b}_1 = 0; \ \text{(m)} \end{array}$$

with similar equations for a_2, b_2 and for a_3, b_3, and for a_4, b_4.

Now $\qquad S_{11} = B + C + D,\ S_{12} = BC + BD + CD,\ S_{13} = BCD,$

with similar meanings for analogous terms. Hence it is easily seen that

$$\begin{array}{l} p(S_{23}, S_{13}) = CD\, p(A, B),\ p(S_{22} \bar{S}_{23}, S_{12} \bar{S}_{13}) = p(A, B), \\ p(S_{21} \bar{S}_{22}, S_{11} \bar{S}_{12}) = p(A, B),\ p(S_{21}, S_{11}) = \bar{C}\bar{D}\, p(A, B). \end{array}$$

Thus

$$p(B, A;\ S_{23}, S_{13};\ S_{22} \bar{S}_{23}, S_{12} \bar{S}_{13};\ S_{21} \bar{S}_{22}, S_{11} \bar{S}_{12};\ S_{21}, S_{11}) = p(A, B);$$

with similar simplifications for the coefficients of $a_1 \bar{b}_1$ and $\bar{a}_1 \bar{b}_1$. Thus equation (m) reduces to the first of equation (32). Similarly for the equations for a_2, b_2 and for a_3, b_3, and for a_4, b_4. Thus in order to find substitutions of the identical group of $\phi(x, y)$, a_1 and b_1 can be chosen to be any pair of roots of the first of equations (32), and then a_2, b_3 and a_3, b_3, and a_4, b_4 must be chosen to be suitable roots of their corresponding equations in (32). Or we may start from a_2, b_2 or from a_3, b_3, or from a_4, b_4.

The identical group of every function is of order greater than one, that is contains substitutions other than the identical substitution.

For if the identical group of $\phi(x, y)$ is of order one the left-hand side of every equation of the set (32) must be a secondary linear prime, so as to give only one pair of roots for a_1, b_1, and one for a_2, b_2, and so on. Now, each of the left-hand sides of equation (32) lacks one term, thus the first of these equations lacks

the term $a_1 b_1$, and so on. Hence [cf. Part I, §3], the left-hand sides can only be secondary linear primes if each remaining coefficient is i.

Thus every equation of the type

$$p(A, B) = i,$$

must hold. But this gives

$$p(\overline{A}, B) = 0, \text{ that is, } A = \overline{B}.$$

Similarly $A = \overline{C}$, $B = \overline{C}$. But these equations are inconsistent. Hence the identical group of $\phi(x, y)$ cannot be of order one.

The identical groups of all members of a congruent family are simply isomorphic.

For, let $\phi(x, y)$ and $\Phi(x, y)$ be two congruent functions; and let the members of the identical group of $\phi(x, y)$ be written T_ϕ, T'_ϕ, etc., and those of the identical group of $\Phi(x, y)$ be written T_Φ, T'_Φ, etc. Also let T be any substitution such that

$$T\phi(x, y) = \Phi(x, y).$$

Then TT_ϕ, TT'_ϕ, ... are evidently such substitutions.

Similarly $T_\Phi T$, $T'_\Phi T$, ... are such substitutions. Again let T_1 be another substitution such that

$$T_1\phi(x, y) = \Phi(x, y),$$

Then $$T^{-1} T_1 \phi(x, y) = T^{-1}\Phi(x, y) = \phi(x, y).$$

Hence $T^{-1} T_1$ is a member of the identical group of $\phi(x, y)$. Accordingly we may write

$$T^{-1} T_1 = T_\phi, \text{ that is, } T_1 = T T_\phi.$$

Similarly $$T_1 = T_\Phi T.$$

Thus each of the sets $T T_\phi$, $T T'_\phi$, ..., and $T_\Phi T$, $T'_\Phi T$, ... includes every substitution which turns $\phi(x, y)$ into $\Phi(x, y)$. Accordingly, by a proper choice of T_ϕ, or of T_Φ, we can always write

$$T T_\phi = T_\Phi T.$$

Hence $$T_\phi = T^{-1} T_\Phi T.$$

Hence the identical groups of $\phi(x, y)$ and of $\Phi(x, y)$ are simply isomorphic. Accordingly it is only necessary to study the structure of the identical group of one member of a congruence family; for instance, that of the canonical form.

For instance, let us investigate the general expression for a substitution of the identical group of the canonical function of the family $(i, i, 0, 0)$. This canonical function is x. Thus if T is the required substitution, we have

$$Tx = x, \; Ty = b_1 xy + b_2 \bar{x}y + \bar{b}_3 xy + b_4 \bar{x}\bar{y}.$$

Hence from equation (12),

$$b_1 b_2 + \bar{b}_1 \bar{b}_2 = 0, \; b_3 b_4 + \bar{b}_3 \bar{b}_4 = 0$$

Thus
$$Ty = xp(\bar{b}_1, y) + \bar{x}p(\bar{b}_3, y); \tag{33}$$

where b_1 and b_3 can be assumed arbitrarily. Thus the general form for a substitution of the group is found.

Now let T and T' be two substitutions of this group, so that

$$\begin{aligned} Tx &= x, \; Ty = b_1 xy + \bar{b}_1 x\bar{y} + b_3 \bar{x}y + \bar{b}_3 \bar{x}\bar{y}, \\ T'x &= x, \; T'y = b_1' xy + \bar{b}_1' x\bar{y} + b_3' \bar{x}y + \bar{b}_3' \bar{x}\bar{y}. \end{aligned}$$

Then

$$T'Ty = \bar{p}(b_1, b_1') xy + p(b_1, b_1') \bar{x}y + \bar{p}(b_3, b_3') \bar{x}y + p(b_3, b_3') \bar{x}\bar{y} = TT'y. \tag{34}$$

Hence the substitutions TT' and $T'T$ are the same. Thus this identical group is Abelian.

Also in equation (34), put $b_1' = b_1$, and $b_3' = b_3$; we find

$$T^2 y = y.$$

Thus $T^2 = T^0$. Hence every substitution of the group is of order two.

The equations satisfied by the coefficients of any substitution of the identical group of the canonical function of the family (S_1, S_2, S_3, S_4), are found from equations (32) to be

$$\left.\begin{aligned} * + S_1\bar{S}_2 a_1 \bar{b}_1 + S_1\bar{S}_3 \bar{a}_1 b_1 + S_1\bar{S}_4 \bar{a}_1 \bar{b}_1 \qquad &= 0, \\ S_1\bar{S}_2 a_2 b_2 + * + S_2\bar{S}_3 \bar{a}_2 b_2 + S_2\bar{S}_4 \bar{a}_2 \bar{b}_2 &= 0, \\ S_1\bar{S}_3 a_3 b_3 + S_2\bar{S}_3 a_3 \bar{b}_3 + * + S_3\bar{S}_4 \bar{a}_3 \bar{b}_3 &= 0, \\ S_1\bar{S}_4 a_4 b_4 + S_2\bar{S}_4 a_4 b_4 + S_3\bar{S}_4 \bar{a}_4 b_4 + * &= 0, \end{aligned}\right\} \tag{35}$$

together with equations (4); and it has been proved that any one pair, such as a_1 and b_1, can be assumed to be any pair of roots of their corresponding equation.

§8.—*Common Subgroups of Identical Groups.*

The identical groups of any two functions have a common subgroup which always includes other substitutions in addition to the identical substitution,

except in the case when the two functions are the two director functions of a substitution.

For let $\phi(x, y)$ and $\Phi(x, y)$ be the two functions, where A, B, C, D are the coefficients of $\phi(x, y)$, and F, G, H, K are the coefficients of $\Phi(x, y)$. Then the coefficients of any substitution common to the two identical groups must satisfy two sets of equations of the type of (32). These two sets can be combined into the single set

$$\left.\begin{aligned}
+p(B, A;\ G, F)\,a_1\bar{b}_1 + p(C, A;\ H, F)\,\bar{a}_1 b_1 + p(D, A;\ K, F)\,\bar{a}_1\bar{b}_1 \quad &= 0,\\
p(A, B;\ F, G)\,a_2 b_2 + \ * \ + p(C, B;\ H, G)\,\bar{a}_2 b_2 + p(D, B;\ K, G)\,\bar{a}_2\bar{b}_2 &= 0,\\
p(A, C;\ F, H)\,a_3 b_3 + p(B, C;\ G, H)\,\bar{a}_3 b_3 + * + p(D, C;\ K, H)\,\bar{a}_3\bar{b}_3 &= 0,\\
p(A, D;\ F, K)\,a_4 b_4 + p(B, C;\ G, K)\,\bar{a}_4 b_4 + p(C, D;\ H, K)\,\bar{a}_4 b_4 + \ * &= 0,
\end{aligned}\right\} \quad (36)$$

and, in addition, equation (4) must be satisfied.

Now, by reasoning in all respects the same as that in the previous article, if there is only one substitution satisfying these conditions, the left-hand side of each of equation (36) is a secondary linear prime. Hence, every equation of the type

$$\bar{p}(B, A;\ G, F) = 0,$$

must hold. But this typical equation is

$$(AB + \bar{A}\bar{B})(FG + \bar{F}\bar{G}) = 0.$$

Hence, by comparison with equation (12), we see that this condition requires that $\phi(x, y)$ and $\Phi(x, y)$ should be a pair of director functions of a substitution. The above proposition can be stated thus: The Identical Group of any function $\phi(x, y)$ which does not belong to the family $(i, i, 0, 0)$ has a subgroup containing more than the one member T^0 in common with the identical group of any other function whatever. The same proposition is true of any function $\phi(x, y)$ which does belong to the family $(i, i, 0, 0)$, except in those cases when the second function also belongs to the same family $(i, i, 0, 0)$, and, in addition, is so related to $\phi(x, y)$ that the two functions are the director functions of a substitution.

It has been proved [cf. Part I, §8, equation* (24)] that we can always write

$$\phi(x, y) = S_4 + S_1(\bar{S}_4 + U_1)\,\phi'(x, y),$$

* A misprint in equation (24), Part I, is here corrected.

where the coefficients of $\phi'(x, y)$ are given by equations (25) of Part I, and the invariants by equations (26) of Part I.

Now $\quad T\phi(x, y) = S_4 + S_1(\overline{S}_4 + U_1)\, T\phi'(x, y).$

Hence, if $\quad T\phi'(x, y) = \phi'(x, y),$

it follows that $\quad T\phi(x, y) = \phi(x, y).$

Hence, the identical group of $\phi'(x, y)$ is a subgroup of the identical group of $\phi(x, y)$. Now, the invariants of $\phi'(x, y)$ are given [cf. equation (26) of §8 Part I] by

$$\left.\begin{aligned} S_1' &= S_1\overline{S}_4 + S_1U_1 + V_1, \\ S_2' &= S_2\overline{S}_4 + S_2U_2 + \overline{S}_1V_2 + \overline{S}_4U_1V_2, \\ S_3' &= S_3\overline{S}_4 + S_3U_3 + \overline{S}_1V_3 + S_4\overline{U}_1V_3, \\ S_4' &= S_4U_4 + \overline{S}_1V_4 + S_4\overline{U}_1V_4, \end{aligned}\right\} \tag{37}$$

and U_1, U_2, U_3, U_4 are the symmetric functions of u_1, u_2, u_3, u_4 and V_1, V_2, V_3, V_4 of v_1, v_2, v_3, v_4.

Also, if $\phi(x, y)$ is the canonical function of the family (S_1, S_2, S_3, S_4), then equations (25) of §8, Part I, become

$$\begin{aligned} A' &= (\overline{S}_1 + S_4\overline{U}_1)\, v_1 + (\overline{S}_4 + u_1)\, S_1, \\ B' &= (\overline{S}_1 + S_4\overline{U}_1)\, v_2 + (\overline{S}_4 + u_2)\, S_2, \\ C' &= (\overline{S}_1 + S_4\overline{U}_1)\, v_3 + (\overline{S}_4 + u_3)\, S_3, \\ D' &= (\overline{S}_1 + S_4\overline{U}_1)\, v_4 + (\overline{S}_4 + u_4)\, S_3. \end{aligned}$$

Hence, in general, $\phi'(x, y)$ does not become the canonical function of (S_1', S_2', S_3', S_4'). But if we make

$$\begin{aligned} u_1 &= U_1, & u_2 &= U_2, & u_3 &= U_3, & u_4 &= U_4, \\ v_1 &= V_1, & v_2 &= V_2, & v_3 &= V_3, & v_4 &= V_4. \end{aligned}$$

which can be done without altering U_1, U_2, U_3, U_4 or V_1, V_2, V_3, V_4, then $\phi'(x, y)$ becomes the canonical function of the family (S_1', S_2', S_3', S_4'). Hence, the identical group of the canonical function of the family (S_1', S_2', S_3', S_4') is a subgroup of the identical group of the canonical function of the family (S_1, S_2, S_3, S_4).

For instance, put

$$\begin{aligned} U_1 &= U_2 = i, & U_3 &= U_4 = 0, \\ V_1 &= V_2 = i, & V_3 &= V_4 = 0. \end{aligned}$$

We deduce that the identical group of the canonical function of the family $(i,\ S_2+\overline{S_1},\ S_3\overline{S_4},\ 0)$ is a subgroup of the identical group of the canonical function of the family $S_1, S_2, S_3, S_4)$.

Now, it is proved in §9 of Part I, that if $S_1 = S_2$ and $S_3 = S_4$, the family contains both linear members and separable members. Also, in this case,

$$S_2+\overline{S_1}=i, \quad S_3\overline{S_4}=0,$$

and the canonical function of the family $(i, i, 0, 0)$ is x. Hence, the identical group of x is a subgroup of the identical groups of the canonical functions of all families which contain both linear and separable members. Thus, from the discussion at the end of the previous paragraph, all these identical groups have a common Abelian subgroup.